Tree Amplitudes and Linearized SUSY Invariants in D=11 Supergravity

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We exploit the tree level bosonic 4-particle scattering amplitudes in $D = 11$ supergravity to construct the bosonic part of a linearized supersymmetry−, coordinate- and gauge-invariant. By differentiation, this invariant can be promoted to be the natural lowest (two-loop) order counterterm. Its existence implies that the perturbative supersymmetry does not protect this ultimate supergravity from infinities, given also the recently demonstrated divergence of its 4-graviton amplitude.

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After the so-called “second string revolution”, D=11 supergravity [1] has regained its well-deserved central role, being a very efficient tool to investigate the (mostly unknown) $M-$theory. In particular, it has been realized how computing loop effects in this last supergravity [2] can shed some light on some (protected) sectors of the $M-$theory effective action [3]. These connections add futher motivation to our quest for higher order on shell SUSY invariants in $D = 11$, whose construction is technically difficult to handle, a tensor calculus being absent.

In this brief review, we will supply (the linearized part of) one such invariant. Our original interest in constructing SUSY was triggered by a more modest, but intriguing question. We wanted to determine unambiguously whether there exist local invariants that can serve as counterterms in loop calculations, at lowest relevant order. This nontrivial exercise has its roots in lower-dimensional SUGRAs, where the existence of invariants is easier to decide. There no miracle seemed to protect this class of theories from diverging, since one is always able to single out a candidate counterterm [4]. However, given all the properties unique to D=11, and the fact that it is the last frontier – a local QFT that is non-ghost (*i.e.*, has no quadratic curvature terms) and reduces to GR – it is sufficiently important not to give up hope too quickly before abandoning D=11 SUGRA and with it all QFTs incorporating GR quickly on non-renormalizability grounds.

The underlying idea is a simple one: the tree level scattering amplitudes constructed within a perturbative expansion of the action are *ipso facto* globally SUSY and linearized gauge invariant. Furthermore, because linearized SUSY means precisely that it does not mix different powers of fields, the 4-point amplitudes taken together form an invariant. Finally, the lowest order bosonic 4-point amplitudes are independent of fermions: virtual fermions never appear at tree level. In order to use this invariant for counterterm purposes, it will first be necessary to remove from it the nonlocality associated with exchange of virtual graviton and form particles .Indeed, the task here will be not only to remove nonlocality but to add sufficient powers of momentum to provide an on-shell invariant of correct dimension to make it an acceptable 2-loop counterterm candidate, this (rather than 1 loop) being the

first possible order (by dimensions) where 4 point amplitudes can contribute. In this way, we will make contact with the conclusive 2-loop results of [2], where it was possible to exhibit the infinity of the 4-graviton component of the invariant. Details of our result can be in [5].

The basis for our computations is the full action of [1], expanded to the order required for obtaining the four-point scattering amplitudes among its two bosons, namely the graviton and the three-form potential $A_{\mu\nu\alpha}$ with field strength $F_{\mu\nu\alpha\beta} \equiv 4\partial_{[\mu}A_{\nu\alpha\beta]}$, invariant under the gauge transformations $\delta A_{\mu\nu\alpha} = \partial_{[\mu}\xi_{\nu\alpha]}$. From the bosonic truncation of this action (omitting obvious summation indices),

$$I_{11}^{B} = \int d^{11}x \left[-\frac{\sqrt{g}}{4\kappa^2}R(g) - \frac{\sqrt{g}}{48}F^2 \right. \tag{1}$$
$$\left. + \frac{2\kappa}{144^2}\epsilon^{1\cdots 11}F_{1\ldots}F_{5\ldots}A_{..11} \right],$$

we extract the relevant vertices and propagators; note that κ^2 has dimension $[L]^9$ and that the (P, T) conserving cubic Chern-Simons (CS) term depends explicitly on κ but is (of course) gravity-independent. The propagators come from the quadratic terms in $\kappa h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ and $A_{\mu\nu\alpha}$; they need no introduction. There are three cubic vertices, namely graviton, pure form and mixed form-graviton that we schematically represent as

$$\begin{aligned} &V_3^g \simeq (\partial h \partial h)h \equiv \kappa T_g^{\mu\nu}h_{\mu\nu}, \quad T_g^{\mu\nu} \equiv G_{(2)}^{\mu\nu}(h), \\ &V_3^{gFF} \equiv \kappa T_F^{\mu\nu}h_{\mu\nu}, \ T_F^{\mu\nu} \equiv F^{\mu}F^{\nu} - \frac{1}{8}\eta^{\mu\nu}F^2, \\ &V_3^F \equiv \kappa A_{\mu\nu\alpha}C_F^{\mu\nu\alpha}, \ C_F^{\rho\sigma\tau} \equiv \epsilon^{\rho\sigma\tau\mu_1\ldots\mu_8}F_{\mu_1\ldots}F_{\ldots\mu_8}. \end{aligned} \tag{2}$$

The form's current C_F and stress tensor T_F are both manifestly gauge invariant. In our computation, two legs of the three-graviton vertex are always on linearized Einstein shell; we have exploited this fact in writing it in the simplified form (2), the subscript on the Einstein tensor denoting its quadratic part in h. To achieve coordinate invariance to correct, quadratic, order one must also include the four-point contact vertices

$$V_4^g \sim \kappa^2(\partial h \partial h)hh, \quad V_4^{gF} = \kappa^2 \frac{\delta I^F}{\delta g_{\alpha\beta}\delta g_{\mu\nu}}h_{\alpha\beta}h_{\mu\nu}$$

when calculating the amplitudes; these are the remedies for the unavoidable coordinate variance of the gravitational stress tensor $T_g^{\mu\nu}$ and the fact that $T_F^{\mu\nu} h_{\mu\nu}$ is only first order coordinate-invariant. The gravitational vertices are not given explicitly, as they are both horrible and well-known [6].

We start with the 4−graviton amplitude, obtained by contracting two V_3^g vertices in all three channels (labelled by the Mandelstam variables (s,t,u)) through an intermediate graviton propagator (that provides a single denominator); adding the contact V_4^g and then setting the external graviton polarization tensors on free Einstein shell. The resulting amplitude $M_4^g(h)$ will be a nonlocal (precisely thanks to the local V_4^g contribution!) quartic in the Weyl tensor[1]. Within our space limitations, we cannot exhibit the actual calculation here; fortunately, this amplitude has already been given (for arbitrary D) in the pure gravity context [6]. It can be shown, using the basis of [7], to be of the form

$$M_4^g = \kappa^2 (4stu)^{-1} t_8^{\mu_1 \cdots \mu_8} t_8^{\nu_1 \cdots \nu_8} \times \tag{3}$$
$$R_{\mu_1\mu_2\nu_1\nu_2} R_{\mu_3\mu_4\nu_3\nu_4} R_{\mu_5\mu_6\nu_5\nu_6} R_{\mu_7\mu_8\nu_7\nu_8} \equiv \frac{L_4^g}{stu},$$

up to a possible contribution from the quartic Euler density E_8, which is a total divergence to this order (if present, it would only contribute at R^5 level). The result (3) is also the familiar superstring zero-slope limit correction to D=10 supergravity, where the $t_8^{\mu_1 \cdots \mu_8}$ symbol originates from the D=8 transverse subspace [8]. [Indeed, the "true" origin of the ten dimensional analog of (3) was actually traced back to D=11 in the one-loop computation of [3].] Note that the local part, L_4^g, is simply extracted through multiplication of M_4^g by stu, which in no way alters SUSY invariance, because all parts of M_4 behave the same way.

In many respects, the form (3) for the 4−graviton contribution is a perfectly physical one. However in terms of the rest of the invariant to be obtained below, one would like a natural

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[1] We do not differentiate in notation between Weyl and Riemann here and also express amplitudes in covariant terms for simplicity, even though they are only valid to lowest relevant order in the linearized curvatures.

formulation with currents that encompass both gravity and matter in a unified way as in fact occurs in e.g. $N=2$, D=4 supergravity [9]. This might also lead to some understanding of other SUSY multiplets. Using the quartic basis expansion, one may rewrite L_4^g in various ways involving conserved BR currents and a closed 4−form $P_{\alpha\beta\mu\nu} = 1/4 R^{ab}_{[\mu\nu} R_{\alpha\beta]ab}$, for example

$$L_4^g = 48\kappa^2 \Big[2B_{\mu\nu\alpha\beta}B^{\mu\alpha\nu\beta} - B_{\mu\nu\alpha\beta}B^{\mu\nu\alpha\beta} +$$

$$P_{\mu\nu\alpha\beta}P^{\mu\nu\alpha\beta} + 6B_{\mu\rho\alpha}{}^{\rho}B^{\mu\sigma\alpha}{}_{\sigma} - \frac{15}{49}(B^{\mu\nu}{}_{\mu\nu})^2 \Big] \tag{4}$$

with $B_{\mu\nu\alpha\beta} \equiv R_{(\underline{\mu}\rho\alpha\sigma}R_{\underline{\nu})}{}^{\rho}{}_{\beta}{}^{\sigma} - \frac{1}{2}g_{\mu\nu}R_{\alpha\rho\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} - \frac{1}{2}g_{\alpha\beta}R_{\mu\rho\sigma\tau}R_{\nu}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\mu\nu}g_{\alpha\beta}R_{\lambda\rho\sigma\tau}R^{\lambda\rho\sigma\tau}$, where ( ) means symmetrization with weight one of the underlined indices.

Let us now turn to the pure form amplitude, whose operative currents are the Chern-Simons $C^F_{\mu\nu\alpha}$ and the stress tensor $T^F_{\mu\nu}$, mediated respectively by the A and graviton propagators; each contribution is separately invariant. We computed the two relevant, $C_F C_F$ and $T_F T_F$, diagrams directly, resulting in the four-point amplitude (see also [10]); $M_4^F = (stu)^{-1}L_4^F = (stu)^{-1}\kappa^2(\partial F)^4$, again with an overall (stu) factor. An economical way to organize L_4^F is in terms of matter BR tensors and corresponding C^F extensions, prototypes being the "double gradients" of $T^F_{\mu\nu}$ and of C^F,

$B^F_{\mu\nu\alpha\beta} = \partial_\alpha F_\mu \partial_\beta F_\nu + \partial_\beta F_\mu \partial_\alpha F_\nu - \frac{1}{4}\eta_{\mu\nu}\partial_\alpha F \partial_\beta F$,

$C^F_{\rho\sigma\tau;\alpha\beta} = \frac{1}{(24)^2}\epsilon_{\rho\sigma\tau\mu_1\cdots\mu_8}\partial_\alpha F^{\mu_1\cdots\mu_4}\partial_\beta F^{\mu_5\cdots\mu_8}$,

where $\partial^\mu B^F_{\mu\nu\alpha\beta} = 0, \quad \partial^\rho C^F_{\rho\sigma\tau;\alpha\beta} = 0.$. From the above equation we can construct L_4^F as

$$L_4^F = \frac{\kappa^2}{36}B^F_{\mu\nu\alpha\beta}B^F_{\mu_1\nu_1\alpha_1\beta_1}G^{\mu\mu_1;\nu_1\nu}K^{\alpha\alpha_1;\beta_1\beta} -$$

$$\frac{\kappa^2}{12}C^F_{\mu\nu\rho;\alpha\beta}C^{F\mu\nu\rho}{}_{\alpha_1\beta_1}K^{\alpha\alpha_1;\beta_1\beta}. \tag{5}$$

The matrix $G^{\mu\nu;\alpha\beta} \equiv \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\nu\alpha}\eta^{\mu\beta} - 2/9\eta^{\mu\nu}\eta^{\alpha\beta}$ is the usual numerator of the graviton propagator on conserved sources. The origin of $K^{\mu\nu;\alpha\beta} \equiv \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\nu\alpha}\eta^{\mu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}$ can be traced back to "spreading" the stu derivatives: for example, in the $s-$channel, *e.g.*, we can write $tu = -1/2K^{\mu\nu;\alpha\beta}p^1_\mu p^2_\nu p^3_\alpha p^4_\beta$; the analogous identities for the other channels can be

obtained by crossing[2]. It is these identities that enabled us to write M_4's universally as $(stu)^{-1}L_4$'s: Originally the M_4 have a single denominator (from the intermediate specific exchange, $s-$, $t-$ or $u-$channel); we uniformize them all to $(stu)^{-1}$ through multiplication of say s^{-1} by $(tu)^{-1}(tu)$. The extra derivatives thereby distributed in the numerators have the further virtue of turning all polarization tensors into curvatures and derivatives of forms, as we have indicated.

The remaining amplitudes are the form "bremsstrahlung" M^{FFFg} and the graviton-form scattering M_4^{Fg}. The M^{FFFg} amplitude represents radiation of a graviton from one of the CS arms, *i.e.*, contraction of the CS and $T^F_{\mu\nu}h^{\mu\nu}$ vertices by an intermediate $A-$line, yielding

$$L_4^{FFFg} = -\frac{\kappa^2}{3} C^F_{\mu\nu\rho;\alpha\beta} C^{RF\mu\nu\rho}{}_{\alpha_1\beta_1} K^{\alpha\alpha_1;\beta_1\beta}, \qquad (6)$$

where the above current $C^{RF}_{\mu\nu\rho;\alpha\beta}$ is given by
$4\partial_\lambda \left(R^{\sigma}{}_{(\alpha}{}^{[\lambda}{}_{\beta)} F_\sigma{}^{\mu\nu\rho]} \right) - \frac{2}{3} R\,{}^{\sigma}{}_{(\alpha}{}^{\lambda}{}_{\beta)} \partial_\lambda F_\sigma{}^{\mu\nu\rho}$.
The off-diagonal current C^{RF} has antecedents in $N = 2$ D=4 theory [9]; it is unique only up to terms vanishing on contraction with C^F. The $M^{Fg}, \sim \kappa^2 R^2 (\partial F)^2$, has three distinct diagrams: mixed $T^F T^g$ mediated by the graviton; gravitational Compton amplitudes $\sim (hh)T_F T_F$ with a virtual $A-$line, and finally the $4-$point contact vertex $FFhh$. The resulting M_4^{Fg} is equal to $(stu)^{-1}L_4^{Fg}$, where

$$L_4^{Fg} = \frac{\kappa^2}{3} \left(\tfrac{1}{4} B^g_{\mu\nu\alpha\beta} B^F_{\mu_1\nu_1,\alpha_1\beta_1} G^{\mu\mu_1;\nu\nu_1} - C^{RF}_{\mu\nu\rho;\alpha\beta} C^{RF\mu\nu\rho}{}_{\alpha_1\beta_1} \right) K^{\alpha\alpha_1;\beta_1\beta}, \qquad (7)$$

up to subleading terms involving traces. The complete bosonic invariant, $L_4 \equiv L_4^F + L_4^g + L_4^{Fg} + L_4^{FFFg}$, is not necessarily in its most unified form, but we hope to return to this point elsewhere.

Finally we discuss the consequences of the very existence of this invariant, for the renormalizability properties of $D = 11$ supergravity. With our space limitation, rewiewing the

[2]It is convenient to define $s \equiv (p_1 \cdot p_2)$, $t \equiv (p_1 \cdot p_3)$, $u \equiv (p_1 \cdot p_4)$, with $p_1 + p_2 = p_3 + p_4$.

general structure of the loop expansion and its possible divergences in $D = 11$ is an impossible task. For this reason, we will limit ourselves to a very brief discussion of the one-loop case and we go directly to the 2−loop analysis. For clarity, we choose to work in the framework of dimensional regularization, in which only logarithmic divergences appear and consequently the local counterterm must have dimension zero.

A generic gravitational loop expansion proceeds in powers of κ^2 (except for odd κ's coming from the CS vertex, see below of). At one loop, one would have $\triangle I_1 \sim \kappa^0 \int dx^{11} \triangle L_1$; but there is no candidate $\triangle L_1$ of dimension 11, since odd dimension cannot be achieved by a purely gravitational $\triangle L_1$, except at best through a "gravitational" $\sim \epsilon\Gamma RRRR$ or "form-gravitational" $\sim \epsilon ARRRR$ CS term (exemplifying the odd power possibility) [11].However,the latter term is in fact forbidden by dimension,since it would require 3 further derivatives but is already of *even* index order. The former violates parity and hence would represent a (necessarily finite) anomaly contribution. The two-loop term would be $\triangle L_2 \sim \kappa^2 \int d^{11}x \triangle L_2$, so that $\triangle L_2 \sim [L]^{-20}$ which can be achieved (to lowest order in external lines) by $\triangle L_2 \sim \partial^{12} R^4$, where ∂^{12} means twelve explicit derivatives spread among the 4 curvatures. There are no relevant 2−point $\sim \partial^{16} R^2$ or 3−point $\sim \partial^{14} R^3$ terms because the R^2 can be field-redefined away into the Einstein action in its leading part (to h^2 order, E_4 is a total divergence in any dimension!) while R^3 cannot appear by SUSY. This latter fact was first demonstrated in D=4 but must therefore also apply in higher D simply by the brute force dimensional reduction argument. So the terms we need are, for their 4-graviton part, L_4^g of (3) with twelve explicit derivatives. The companions of L_4^g in L_4^{tot} will simply appear with the same number of derivatives. It is easy to see that the additional ∂^{12} can be inserted without spoiling $SUSY$; indeed they appear as naturally as did multiplication by stu in localizing the M_4 to L_4: for example, ∂^{12} might become, in momentum space language, $(s^6 + t^6 + u^6)$ or $(stu)^2$. This establishes the structure of the 4−point local counterterm candidate.

Before the present construction of the complete counterterm was achieved, the actual coefficient of its 4-graviton part was computed [2] by a combination of string-inspired and

unitarity techniques. Their final result was

$$\mathcal{M}_4^{g\text{twoloop},\ D=11-2\epsilon}|_{\text{pole}} = \left(\frac{\kappa}{2}\right)^6 \times (stuM_4^{\text{g tree}})\times$$

$$\frac{1}{48\epsilon\ (4\pi)^{11}}\frac{\pi}{5791500}\Big(438(s^6+t^6+u^6)-53s^2t^2u^2\Big)\,,$$

where $(stu)M_4^{\text{g tree}}$ is given in (3). The extension of this expression into a counterterm lagrangian for the rest of the bosonic sector was not presented in [2], but is effectively completed here. For detail of [2], we refer the reader to the review by Bern included in this same volume. One final comment: nonrenormalizability had always been a reasonable guess as the fate of D=11 supergravity. The opposite guess, however, that some special (M−theory related?) property of this "maximally maximal" model might keep it finite could also have been reasonably entertained a priori, so this was an issue worth settling.

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[1] E. Cremmer, B. Julia and J. Scherk, *Phys. Lett. B* **79** (1978) 409.

[2] Z. Bern, L. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *Nucl. Phys. B* **530** (1998) 401.

[3] M. B. Green, M. Gutperle and P. Vanhove, *Phys. Lett. B* **409** (1997) 177; M. B. Green, *Nucl. Phys. Proc. Suppl.* **68** (1998) 242; M. B. Green, H. Kwon and P. Vanhove, *Two loops in Eleven-Dimensions*, hep-th/9910055.

[4] for a review see: S. Deser, *Infinities in Quantum Gravities*, gr-qc/9911073.

[5] S. Deser and D. Seminara, *Phys. Rev. Lett.* **82** (1999) 2435; and *Tree Amplitudes and Two-loop Countertetrms in $D=11$ Supergravity*, BRX-TH-456 to be submitted.

[6] S. Sannan, *Phys. Rev. D* **35** (1987) 1385.

[7] S. A. Fulling et al., *Class. Quant. Grav.* **9** (1992) 1151.

[8] J. H. Schwarz, *Phys. Rept.* **89** (1982) 223.

[9] S. Deser and J. H. Kay, *Phys. Lett. B* **76** (1978) 400.

[10] J. Plefka, M. Serone and A. Waldron, *JHEP9811* (1998) 010.

[11] M. J. Duff and D. Toms, in "*Unification of the fundamental particle interactions II*", (Plenum Press, New York, 1982).